Constant Capacity Multicommodity Fixed-Charge Network Design Problems

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Abstract

Capacitated multicommodity fixed-charge network design problems can be used to model long-term planning for transportation, telecommunications, and other service operations. This paper studies a minimum cost constant capacitated multicommodity network design model with integral flow requirement, where arcs and routes are selected to meet the demand for different commodities. Using three-dimensional matching, we show that this problem is NP-complete even when the constant arc capacity requirement is as small as 2. We propose different classes of inequalities to strengthen its arc flow formulation. The model and methods developed in this paper can be extended to the planning in transportation, telecommunication, and other sectors.

1. Introduction

Service network design has long been a fundamental problem in long-term planning for transportation, supply chain, telecommunications, and other services used to transport material, information, energy, and other resources. These networks require annual investments of hundreds of billions of dollars. Therefore, significant cost savings can accrue by optimizing network topological design. In a typical service network design problem, we are given a network with candidate arcs and a list of pairs of network nodes, each associated with end-to-end service demand. Such a problem aims to minimize the design cost associated with selecting the network design arcs and the routing cost associated with sending the commodities on these arcs while ensuring that end-to-end service requirements are met. Motivated by the fact that each arc in the service network has a finite capacity, we incorporate the arc capacity limit in the service network design problem. Due to the complex interactions among the design and routing decisions, even the simplest versions of the service network design problem are hard to solve. The goal of this paper is to develop effective optimization-based methods to support service network design decisions.

This paper proposes a fixed-charge multicommodity network design problem with a constant arc capacity limit (NDCL). The problem is formulated as a large-scale integer programming model that generalizes previous constant capacity fixed-charge network flow problems. In the NDCL problem, we need to decide which arcs should be included in the design and how to route each commodity on chosen arcs, subject to end-to-end service requirements and the arc capacity, to minimize the sum of fixed arc costs and commodity routing costs. The NDCL problem is hard to solve since some special cases of the NDCL problem are well-known as computationally intractable (NP-hard). Further, we show that the NDCL problem is NP-hard even when all arcs have the same capacity limit of 2.

There has been quite extensive research work devoted to network design problems (see, e.g., Crainic (2000); Magnanti and Wong (1984); Minoux (1989)). Natu and Shu-Cherng (1997) study the point-to-point connection problem, which is to find a subset of arcs with minimal total selecting cost connecting a fixed number of source-destination pairs. Goemans and Williamson (1995) presents a general approximation technique for a large class of graph problems, including the point-to-point connection problem. A generalization of the point-to-point connection problem is the Steiner forest problem, which is to find the cheapest subgraph to connect the given terminal pairs. Gassner (2010) discuss the Steiner forest problem and show that the Steiner forest problem is strong NP-hard on graphs with treewidth 3. Gupta and Kumar (2015) analyze the greedy algorithm for the Steiner forest problem and use it to give new, simpler constructions of cost-sharing schemes for the Steiner forest. Hubert Chan et al. (2018) achieve a (randomized) polynomial-time approximation scheme (PTAS) for the Steiner forest problem in doubling metrics. Other approximations for Buy-at-bulk network design literature consider the node weighted or online version (see, e.g., Chekuri et al. (2007, 2010);

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Chakrabarty et al. (2018)).

Another special case of the NDCL problem is the buy-at-bulk problem, in which the routing costs are included. Salman et al. (1997) presents an approximation algorithm for minimum cost network design that routes all the demands at the sources to a single sink. Gupta et al. (2003) give a simple constant-factor approximation algorithm for the single-sink buy-at-bulk network design problem. Andrews (2004) analyze the hardness of approximation algorithm for the non-uniform buy-at-bulk network design problem. Charikar and Karagiozova (2005) study the multicommodity buy-at-bulk network design problem in which we seek to design a network that satisfies the demands between terminals from a given set of source-sink pairs. Antonakopoulos (2011) develop two approximation algorithms for directed buy-at-bulk network design in the non-uniform cost model. Antonakopoulos et al. (2011) consider approximation algorithms for buy-at-bulk network design, with the additional constraint that demand pairs be protected against a single edge or node failure in the network.

The most related to the NDCL problem is the Multicommodity Capacitated Fixed-Charge Network Design, where the difference is that the routing variables are continuous. Atamturk (2002) provides an analysis of capacitated network design cut–set polyhedra. Costa et al. (2009) compare three sets of inequalities: Benders, metric, and cutset inequalities, which have been widely used in solving multicommodity capacitated network design problems. Frangioni and Gendron (2009) study 0–1 reformulations of the multicommodity capacitated network design problem, which is usually modeled with general integer variables to represent design decisions on the number of facilities to install on each arc of the network. Chouman et al. (2017) improve the mixed-integer programming formulation of the multicommodity capacitated fixed-charge network design problem by incorporating valid inequalities into a cutting-plane algorithm. Other streams of the capacitated network design literature utilize dual ascent, Lagrangian relaxation, or local search approaches (see, e.g., Herrmann et al. (1996); Holmberg and Yuan (2000); Katayama et al. (2009); Rodríguez-Martín and Salazar-González] (2010)).

The rest of this paper is organized as follows. In Section 2, we define the NDCL problem, formulate it as an integer programming problem, and show the NP-hardness of the NDCL problem with an arc capacity of 2. In Section 3, we develop inequalities to strengthen the NDCL model and provide examples to support their effectiveness. Section 4 concludes the paper.

2. Model and Hardness

2.1. Problem Definition

The NDCL problem is defined over a given directed graph $G = (N, A)$, where nodes $i \in N$ representing origins, destinations, or transshipment points for traffic, and arcs $a \in A$ representing available interconnections we can select. K denotes the set of all commodities. An ordered pair $(s_k, t_k) \in N \times N$ is called a demand pair; furthermore, s_k and t_k are the origin and destination of commodity $k \in K$. We normalize the demand for each commodity to one unit and require transporting it on a single origin-to-destination route.

Let $i(a)$ and $j(a)$ denote the tail node and head node of arc a, respectively. The set of outgoing arcs from node i is defined as $A^+(i) = \{a \in A : i(a) = i\}$ and the set of incoming arcs into node j is defined as $A^-(j) = \{a \in A : j(a) = j\}$. The two non-negative cost coefficients f_a and c_a for every arc $a \in A$ represent the installation cost and routing cost, respectively. In the telecommunication context, installing a link with a certain technology (e.g., fiber optic, cellular) incurs the fixed cost f_a. The routing costs represent operational costs. In reality, we can only route a finite number of commodities over an arc. This paper assumes the arc capacity limit L is fixed and uniform for each arc in the network.

The NDCL problem aims to minimize the total arc fixed costs and routing costs by selecting arcs of the given network and routing each commodity on a feasible route based on the selected arcs. To model this problem, we introduce two sets of binary variables, the design and routing decisions, respectively. The design variables z_a, for all arcs $a \in A$, represent the choice of arcs to be included in the network design solution. z_a is one if arc a is selected, and zero otherwise. The routing variable x_a^k, for arc $a \in A$ and commodity $k \in K$, denotes the commodity routing decisions; x_a^k equals one if commodity k is routed on arc a, and zero otherwise. Using the above variables, the NDCL problem can be formulated as Model *NDCL*:

$$\min \sum_{a \in A} f_a z_a + \sum_{a \in A} c_a \left(\sum_{k \in K} x_a^k \right) \tag{1}$$

s.t.

$$\sum_{a \in A^-(i)} x_a^k - \sum_{a \in A^+(i)} x_a^k = \begin{cases} -1 & \text{if } i = s_k, \\ 1 & \text{if } i = d_k \text{ , } \forall k \in K, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

$$\sum_{k \in K} x_a^k \le L z_a, \forall a \in A, \tag{3}$$

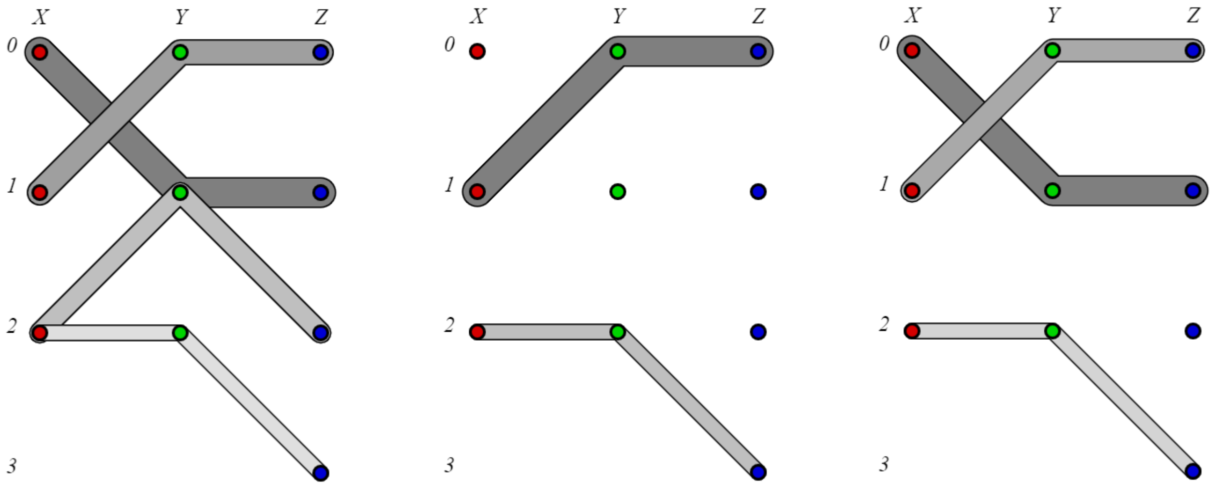
$$x_a^k \in \{0,1\}, \forall a \in A, \forall k \in K, \tag{4}$$

$$z_a \in \{0,1\}, \forall a \in A. \tag{5}$$

The objective function Eq. [1] consists of the total fixed cost for selecting arcs on the given network and the total routing cost incurred by the commodities over all the arcs. The flow conservation constraints Eq. [2] ensure that the arc routing variables define an origin to destination route for each commodity. Constraints Eq. [3] link the design and arc routing variables by specifying that we can route at most L commodities arc a if it's selected in the network design. In many applications, especially in transportation, the flow of a commodity is restricted to run through a single path in the network. Similarly, the status of the arc in the network is also often binary, either in use or not in use. Therefore, we require both the design and flow variables to be binary. We explore the case where the arc capacity is two, and show that the NDCL problem is NP-complete even in this simple setting.

2.2. Computational Hardness

We first introduce the well-known 3-dimensional matching problem (e.g., Karp (1972); Korte and Vygen (2007)), which will be used in the following reduction. Let X, Y, and Z be finite, disjoint sets, and let T be a subset of $X \times Y \times Z$. That is, T consists of triples (x, y, z) such that $x \in X$, $y \in Y$, and $z \in Z$. Now $M \subseteq T$ is a 3-dimensional matching if the following holds: for any two distinct triples $(x_1, y_1, z_1) \in M$ and $(x_2, y_2, z_2) \in M$, we have $x1 \neq x2$, $y1 \neq y2$, and $z1 \neq z2$.



(a) Input. (b) Feasible solution. (c) Optimal solution.

Figure 1: An instance of 3-dimensional matchings.

Given an instance of 3-dimensional matching, we construct a corresponding instance of the NDCL problem with an arc capacity limit of two. For every triple (x_i, y_j, z_k) in the original 3-dimensional matching problem, we construct a simple path (x_i, y_j, z_k, w_i), where $w_i \in w$ is the endpoint of the simple path. As shown in Fig 2, we need to construct a new arc between sets Z and W for each triple. Note that each $w_i \in W$ corresponds to an $x_i \in X$; thus, $|W| = |X|$. Assume now we have a set of real commodities R_1 whose origin and destination pairs are (x_i, w_i), $i = 1, ..., |X|$. Assume we also have two sets of dummy commodities T_1 and T_2, where $|T_1| = |Y|$, $|T_2| = |Z|$. The origin node of each dummy commodity in T_1 is assigned to be a unique node in Y, the origin node of each dummy commodity in T_2 is assigned to be a unique node in Z, *i.e.*, no two dummy commodities share the same origin. The destination node for dummy commodities in T_1 is D_1, the destination node for dummy commodities in T_2 is D_2. We connect all the nodes in Z with D_1 and all the nodes in W with D_2 using zero cost dummy arcs. The number of dummy arcs between z_k and D_1 is $\lceil |A^-(z_k)|/L \rceil$, the number of dummy arcs between w_i and D_2 is $\lceil |A^-(w_i)|/L \rceil$. Assume both the fixed and all routing costs on arcs between Y and Z are 1, the fixed and routing costs on arcs between Z and W are also 1. All the other arcs in the graph have zero fixed and routing costs. Since the fixed and routing cost total for routing any dummy commodity is two anyway, we only count the costs for routing real commodities to simplify the analysis. (The total cost of routing dummy commodities will remain unchanged when we change the path of the real commodities.) Assume there is a private path between x_i and d_i for each real commodity, $i = 1, ..., |X|$. The fixed cost of using the private path is 1 and the routing cost on the private path is $1 + \epsilon$, where $0 < \epsilon < 1$. Here we show that for the optimal

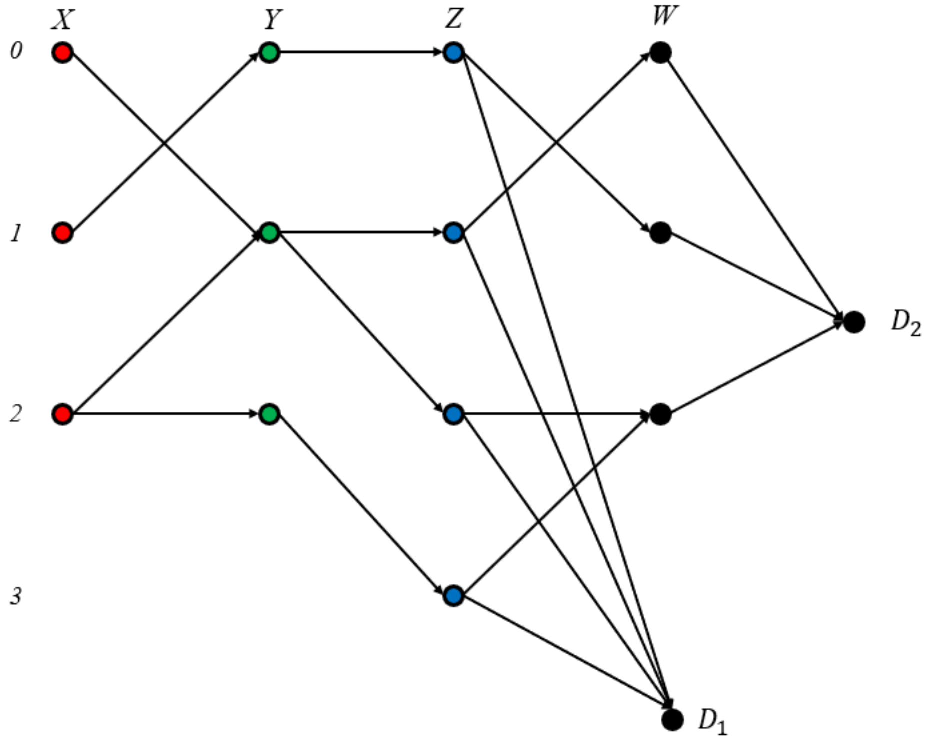


Figure 2: An instance of NDCL problem extended from the 3-dimensional matching.

solution of the NDCL problem with an arc capacity limit of 2, either a real commodity goes over a unique triple in the original 3-dimensional matching problem or through its private path (remains unmatched in the 3-dimensional matching setting).

Lemma 1. *No path of two real commodities will share the same node in Z in the optimal solution.*

Proof. Assume the statement is not true. There are paths of two real commodities that will share the same node in Z in the optimal solution. Since there is one and only one dummy commodity at each node in Z, and every pair of two real commodities has different destinations, if there are paths of two commodities that share the same node z in Z, only one of them can go with the dummy commodity. Assume the two commodities are commodity i and j. We let commodity i go with the dummy commodity since the total cost will be the same no matter which one goes with the dummy commodity at z. If j goes with a dummy commodity or a real commodity over an arc between Y and Z, we fixed the paths of other commodities, and move j to its private path, the reduced cost will be at least 3, and the induced cost is only $2 + \epsilon$. Since $2 + \epsilon < 3$, the total cost will decrease if we move commodity j to its private path, which contradicts the assumption. Therefore, no two real commodities will share the same node in Z in the optimal solution. □

Lemma 2. *No path of two real commodities will share the same node in Y in the optimal solution.*

Proof. By lemma 1, in the optimal solution every pair of real commodities will go to different nodes in Z, so we can use the same argument in lemma 1, no path of two real commodities will share the same node in Y in the optimal solution. □

Theorem 1. *No path of two real commodities will share the same node in the optimal solution.*

Proof. Combine the result of lemma 1 and 2 and the fact every pair of real commodities has a different origin and destination. We can conclude that no path of two real commodities will share the same node in the optimal solution. □

Therefore, in the optimal solution of the NDCL problem with arc capacity limit of 2 as shown in Fig 2, each commodity will go over either unique path of the 3-dimensional matching problem or go through its private path. Since we can attach the fixed cost selecting arcs in the network to dummy commodities, the cost for the dummy commodities will be constant for any solution of the above designed NDCL problem. We denote the cost for the dummy commodities as c_d, where $c_d = 2|Y| + 2|Z|$. So the total cost of the optimal solution will be

$$c = c_d + 2m + (|X| - m)(2 + \epsilon) = 2(|X| + |Y| + |Z|) + \epsilon(|X| - m),$$

where m is the number of paths that contain a unique triple in the optimal solution. Minimizing the total cost c is equivalent to maximize the number of triples for the 3-dimensional matching problem.

Theorem 2. *The decision version of the NDCL problem with an arc capacity limit of 2 is NP-complete.*

Proof. Note that the 3-dimensional matching problem can be reduced to the NDCL problem with an arc capacity limit of 2, and the 3-dimensional matching problem is a well-known NP-complete problem. The NDCL problem with an arc capacity limit of 2 is also NP-complete. □

This proof can be generalized to any fixed arc capacity limit. If the arc capacity limit $L \geq 2$, the NDCL problem is NP-hard even if the arc capacity limit is fixed and uniform over the given network topology, *i.e.*,

Theorem 3. *The decision version of the NDCL problem with arc capacity limit of $L(L \geq 2)$ is NP-complete.*

3. Strengthening the NDCL Model

In this section, we present different classes of inequalities to strengthen the formulation [NDCL]. We also give some examples to illustrate how the inequalities eliminate some fractional solutions of [NDCL-LP] and strictly raise the optimal LP value.

3.1. Strong Inequalities for Single Arc Design Relaxation

Now we consider the unsplittable flow arc set on every single arc. Follow the notations in Atamturk and Rajan (2002), the feasible unsplittable flow arc set over an arc can be represented as

$$F_u = \left\{ \left(x_a^k\right)_{k\in K}, z_a | \sum_{k\in K} x_a^k \leq L z_a, x_a^k \in \{0,1\}, k \in K, z_a \in \{0,1\} \right\}. \tag{6}$$

The convex hull of solutions in F_u is defined as

$$conv(F_u) = \left\{ \left(x_a^k\right)_{k\in K}, z_a | \sum_{k\in K} x_a^k \leq L z_a, x_a^k \in [0,1], k \in K, z_a \in [0,1] \right\}. \tag{7}$$

The resulting relaxation decomposes by arc, it's also called *single-arc design relaxation* in Magnanti et al. (1993). Then, the following *strong inequalities* (SI) are valid for $conv(F_u)$ and will cut off some fractional solutions,

$$x_a^k \leq z_a, \quad \forall k \in K. \tag{8}$$

It can be verified that the *strong inequalities* are facet-defining for $conv(F_u)$, see Atamturk and Rajan (2002).

Proposition 1. *The strong inequalities 8 are facet-defining inequalities of* $conv(F_u)$.

Adding all the SIs for all the arcs to [NDCL-LP] will significantly improve the lower bound. However, the number of SI is $|A||K|$. It yields an extremely large model if we add all of the SI to the LP relaxation. That will frequently exhibit degeneracy with a high computational cost. In fact, only a small number of SI is necessary and will yield efficient result. In our cutting plane algorithm, only the most violated SI is added to the LP relaxation for each run.

3.2. Cover inequality

Let $S \subset N$ be a non-empty subset of N, and $\bar{S} = N \backslash S$ be its complement, the corresponding cut-set is defined as $(S, \bar{S}) = \{a \in A | i(a) \in S, j(a) \in \bar{S}\}$. We denote the associated commodity subset as $K(S, \bar{S}) = \{k \in K | s_k \in S, t_k \in \bar{S}\}$. Further, we use $d_{S,\bar{S}} = |K(S, \bar{S})|$ to denote the total demand that must flow from S to $\bar{S}$. There should be enough capacity on the arcs of the cut-set $(S, \bar{S})$ to satisfy the total demand requirement, we can obtain the *Single-Cut-Set Relaxation*, which is

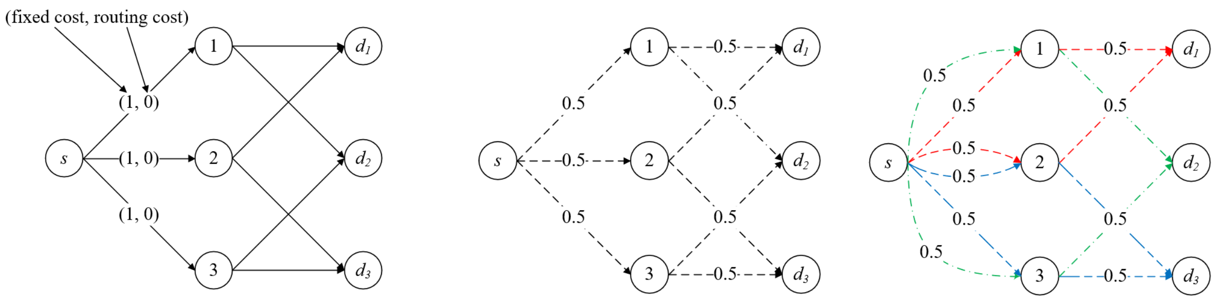
$$\sum_{a \in (S,\bar{S})} L z_a \geq d_{S,\bar{S}}. \tag{9}$$

Using the integrity of z_a for all $ain(S, \bar{S})$, we can derive the *Cover Inequality*

$$\sum_{a \in (S,\bar{S})} z_a \geq \left\lceil \frac{d_{S,\bar{S}}}{L} \right\rceil. \tag{10}$$

Proposition 2. *The Cover Inequality 10 is valid and tightens formulation 1.*

Example for *Cover inequality*. The example in Figure 3 and 4b show the effectiveness of the *Cover Inequality*. In this example, there are three commodities k_1, k_2, and k_3. All the three commodities have the same origin s with respective destinations d_1, d_2, and d_3. The fixed arc cost for arcs $(s, 1)$, $(s, 2)$, and $(s, 3)$ is 1, the routing cost is 0. The fixed arc and routing costs are 0 for other arcs (not shown in the Figure 3a), and the capacity limit we consider is $L = 2$. Figure 3b and 3c illustrate the optimal LP solution. Consider the partition as $S = \{s\}$, $\bar{S} = \{1, 2, 3, d_1, d_2, d_3\}$, the corresponding cut-set is $(S, \bar{S}) = \{(s, 1), (s, 2), (s, 3)\}$. The demand from S to $\bar{S}$ is $|d_{S,\bar{S}}| = 3$. We then get the *Cover Inequality* $z_{s,1} + z_{s,2} + z_{s,3} \geq \lceil 3/2 \rceil = 2$, which is violated by the LP solution in 3. Adding this *Cover inequality*, we will get the optimal integer solution. Note that the *Cover Inequality* is especially useful for the set cover type problems.

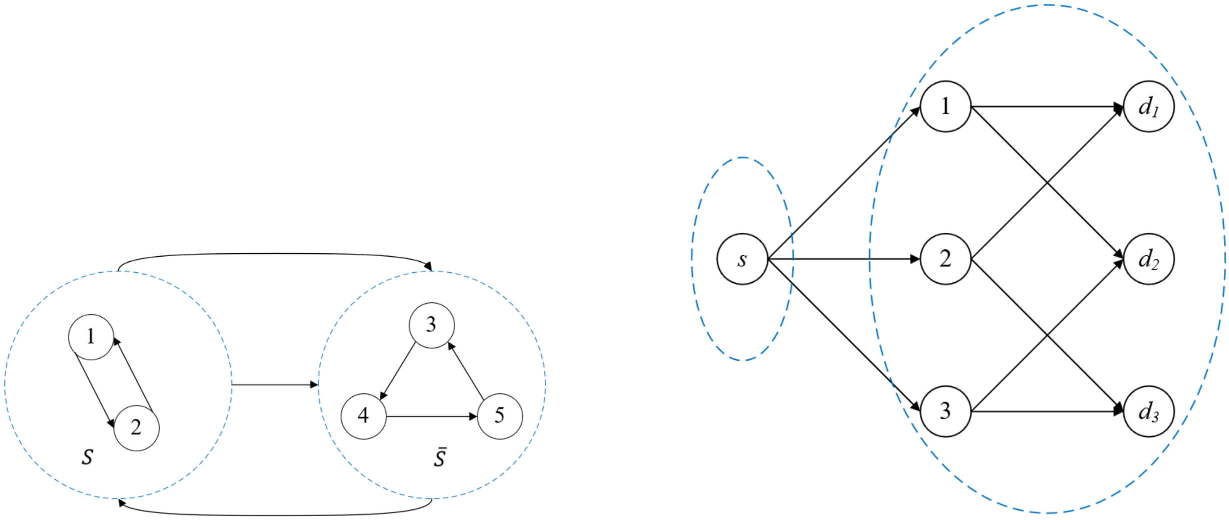


(a) Network structure. (b) Optimal LP solution (z). (c) Optimal LP solution (x).

Figure 3: Example for *Cover Inequality*.

3.3. Two-partition flow cover inequality

One generalization of the *Cover Inequality* is the *Flow Cover Inequality*. Given a two-partition of the network G, *i.e.*, S and $\bar{S}$, for any $a \in A$, let $x_a = \sum_{k \in K} x_a^k$, $d_{S,\bar{S}} = |K(S, \bar{S})|$, $d_{\bar{S},S} = |K(\bar{S}, S)|$. An example of two-partition of



(a) A two-partition of the network.

(b) Partition for the cover inequality.

the network is shown in Figure 4a. By summing the flow conservation Equations 2 for all $i \in S$ and $k \in K$, we obtain

$$\sum_{a\in(S,\bar{S})} x_a - \sum_{a\in(\bar{S},S)} x_a = d_{S,\bar{S}} - d_{\bar{S},S}. \tag{11}$$

Since $d_{\bar{S},S} \geq 0$, we can get the *single-cut-set flow relaxation* by relaxing the equality in Equation 11, SCF_S. The feasible set of SCF_S is defined as

$$F(SCF_S) = \Big\{ (x_a, z_a)_{a\in(S,\bar{S})\cup(\bar{S},S)} | x_a \leq L z_a, \sum_{a\in(S,\bar{S})} x_a^M - \sum_{a\in(\bar{S},S)} x_a \leq d_{S,\bar{S}}, \\ x_a, z_a \in \{0,1\}, a \in (S,\bar{S}) \cup (\bar{S},S) \Big\}. \tag{12}$$

This relaxation reduces the NDCL problem to the single-node fixed-charge flow problem. Padberg et al. (1985) studied the convex hull of $F(SCF_S)$.

A set $C \subseteq (S,\bar{S})$ is called a flow cover if $\lambda = L|C| - d_{S,\bar{S}} \geq 0$. Moreover, the flow cover C is minimal if $\lambda < L$. Note that in a minimal flow cover, $\lambda = \left\lceil \frac{d_{S,\bar{S}}}{L} \right\rceil L - d_{S,\bar{S}}$. Given a flow cover $C \subseteq (S,\bar{S})$, the flow cover inequality is defined as

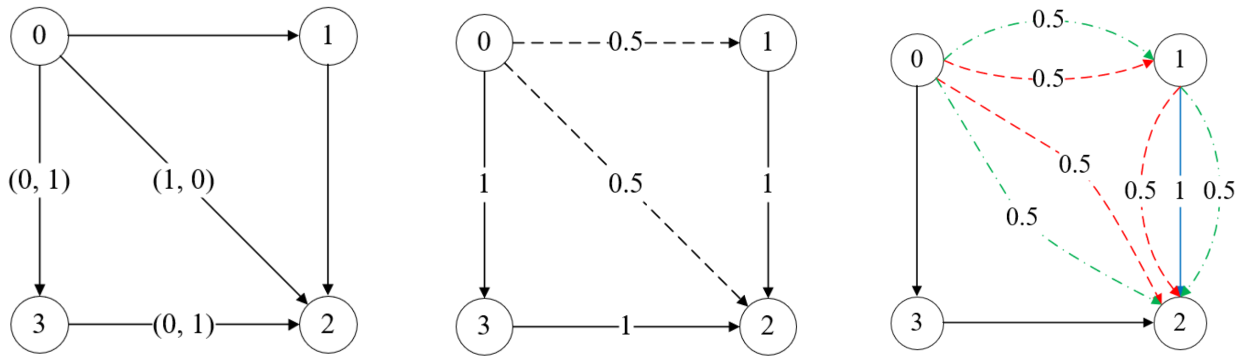
$$\sum_{a\in(S,\bar{S})} \left[x_a + \rho(1 - z_a)\right] - \sum_{a\in(\bar{S},S)} \min\left\{x_a, (L-\rho)z_a\right\} \leq d_{S,\bar{S}}, \tag{13}$$

where $\rho = (L - \lambda)_+$. The lifted flow cover inequality is

$$\sum_{a\in(S,\bar{S})} \left[x_a + \rho(1 - z_a)\right] - \sum_{a\in(\bar{S},S)} \min\left\{x_a, (L-\rho)z_a\right\} + \sum_{(S,\bar{S})\setminus C} \max\left\{0, x_a - \rho z_a\right\} \leq d_{S,\bar{S}}. \tag{14}$$

When applying the flow cover inequalities (13) and (14), many nodes collapsed to one. The internal structure of node partitions S and $\bar{S}$ is ignored.

Example for *two-partition flow cover inequalities*. Consider the network depicted in Figure 6, and assume there are three commodities, $s_1 = s_2 = 0$, $s_3 = 1$, $d_1 = d_2 = d_3 = 2$. Assume that the arc capacity is 2, with the *cover*



(a) Network structure. (b) Optimal LP solution (z). (c) Optimal LP solution (x).

Figure 5: Example for *Two-partition flow cover inequality*.

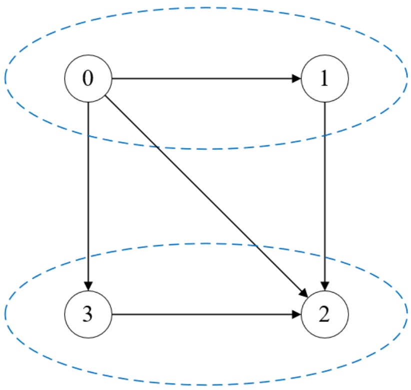


Figure 6: Partition for the *Two-partition flow cover inequality*.

inequalities

$$z_{0,2} + z_{1,2} + z_{3,2} \geq 2, z_{0,2} + z_{1,2} + z_{0,3} \geq 2,$$

one of the optimal LP solutions is

$$\begin{aligned} x^1_{0,1} = x^1_{1,2} = x^2_{0,1}, = x^2_{1,2} &= 0.5, \\ x^1_{0,2} = x^2_{0,2} = z_{0,2} &= 0.5, \\ x^3_{1,2} = z_{1,2} &= 1, \\ z_{0,3} = z_{3,2} &= 1. \end{aligned}$$

Let $S = \{0, 1\}$, and $C = \{(0, 2), (1, 2)\}$. Then, $d_{S,\bar{S}} = 3$, $\lambda = \rho = 1$, the flow cover inequality is

$$x_{1,2} + \left(1 - z_{1,2}\right) + x_{0,2} + \left(1 - z_{0,2}\right) = 3.5 > 3.$$

The flow cover inequality is violated in this case. Adding the flow cover inequality

$$x_{1,2} + \left(1 - z_{1,2}\right) + x_{0,2} + \left(1 - z_{0,2}\right) \leq 3$$

will yield the optimal IP solution.

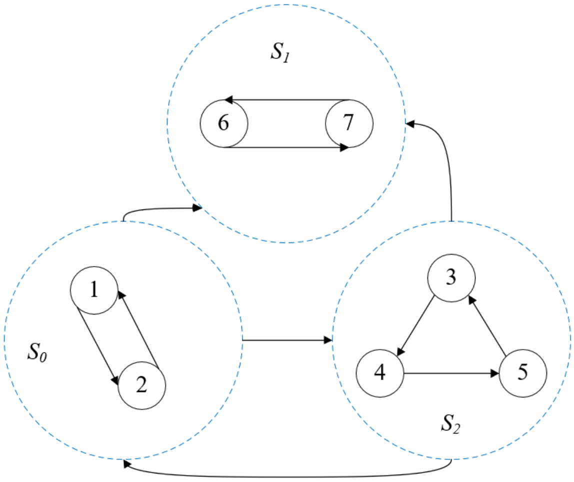


Figure 7: A three-partition of the network.

3.4. Three-partition flow cover inequality

In addition to the two-partition flow cover inequality, Atamtürk et al. (2016) proposed a three-partition flow cover inequality for constant capacity fixed-charge network flow problems. A three-partition of the network is shown in Figure 7. In the three partition, the nodes are divided into three sets S_0, S_1 and S_2. We use (S_i, S_j) to denote the arc set, where $i(a) \in S_i$, $j(a) \in S_j$, and the associated commodities are $K(S_i, S_j)$, $0 \leq i, j \leq 2$.

3.4.1. Three-partition flow cover inequality with two destinations

To be consistent with Atamtürk et al. (2016), we use S_0 as the source nodes, S_1 and S_2 are the destination nodes. Without loss of generality, we assume that $d_{S_1,S_0} = d_{S_1,S_2} = d_{S_2,S_1} = d_{S_2,S_0} = 0$.

For $C_1 \subseteq (S_0, S_1)$, $C_2 \subseteq (S_0, S_2)$, and $C_{12} \subseteq (S_1, S_2)$, the set $C = C_1 \cup C_2 \cup C_{12}$ is a three-partition flow cover if

1. $\lambda_1 = L|C_1| - d_{S_0,S_1} \geq 0$,
2. $\lambda_2 = L|C_2 \cup C_{12}| - d_{S_0,S_2} \geq 0$,
3. $\lambda = L|C_1 \cup C_2| - \left(d_{S_0,S_1} + d_{S_0,S_2}\right) \geq 0$.

Furthermore, the three-partition flow cover is minimal if

4. $\lambda_2 < L$ and $\lambda \leq L$.

Given a minimal three-partition flow cover, the three-partition flow cover inequality is defined as

$$\sum_{i=1,2}\sum_{a\in C_i}\left[x_a + \rho_i(1-z_a)\right] - \sum_{i=1,2}\sum_{a\in(S_i,S_0)}\min\left\{x_a, (L-\rho_i)z_a\right\} + \sum_{i=1,2}\sum_{a\in(S_0,S_i)\backslash C_i}\max\left\{x_a - \rho_i z_a, 0\right\}$$
$$+ \sum_{a\in C_{12}}(\rho_2-\rho_1)(1-z_a) - \sum_{a\in(S_1,S_2)\backslash C_{12}}\min\left\{x_a, (\rho_2-\rho_1)z_a\right\} + \sum_{a\in(S_2,S_1)}\max\left\{0, x_a + (\rho_2 - \rho_1 - L)z_a\right\} \leq d_{S_0,S_1} + d_{S_0,S_2}, \quad (15)$$

where

$$(\rho_1, \rho_2) = \begin{cases} \big(L-\lambda, L-\lambda+(\lambda-\lambda_2)_+\big) \\ \quad \text{(Type 1 three-partition flow cover)} \\ \big((\lambda_2-\lambda)_+, L-\lambda_2+(\lambda_2-\lambda)_+\big) \\ \quad \text{(Type 2 three-partition flow cover)} \end{cases}$$

Example for three-partition flow cover inequalities. Consider the network in Figure 8, assume there are three commodities, $s_1 = s_2 = s_3 = 0$, $d_1 = d_2 = 1$, $d_3 = 2$. One of the optimal LP solutions is

$$x^1_{0,2} = x^1_{2,1} = x^2_{0,2} = x^2_{2,1} = 0.5,$$
$$x^1_{0,1} = x^2_{0,1} = 0.5, x^3_{0,2} = 1,$$
$$z_{0,1} = z_{0,2} = 1, z_{2,1} = 0.5.$$

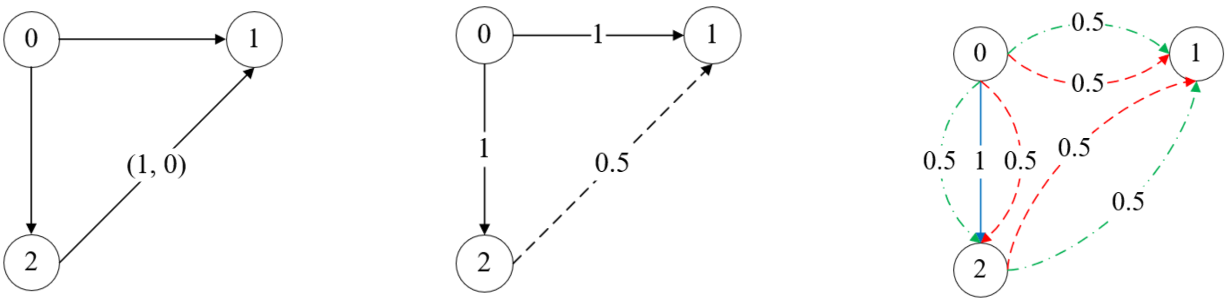
It's easy to check that neither the cover or flow cover inequalities is not violated. Let $S_0 = \{0\}$, $S_1 = \{1\}$, $S_2 = \{2\}$, $C_1 = \{(0,1)\}$, $C_2 = \{(0,2)\}$, $C_{1,2} = \emptyset$. Therefore, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda = 1$. We use the type 2 three-partition flow cover inequality, which is

$$x_{0,1}+\rho_1(1-z_{0,1}) + x_{0,2} + \rho_2(1-z_{0,2}) + \max\left\{0, x_{2,1} + (\rho_2-\rho_1-L)z_{2,1}\right\} = 3.5 > 3,$$

where $(\rho_1, \rho_2) = (0, 1)$. The type 2 three-partition flow cover inequality is violated by the optimal LP solution. Adding this three-partition flow cover inequality, we can get the optimal solution.

Proposition 3. *The* three-partition flow cover inequality with two destinations *is valid for the NDCL problem, and strengthens formulation 1.*

The proof of this proposition is the same as in Atamtürk et al. (2016).



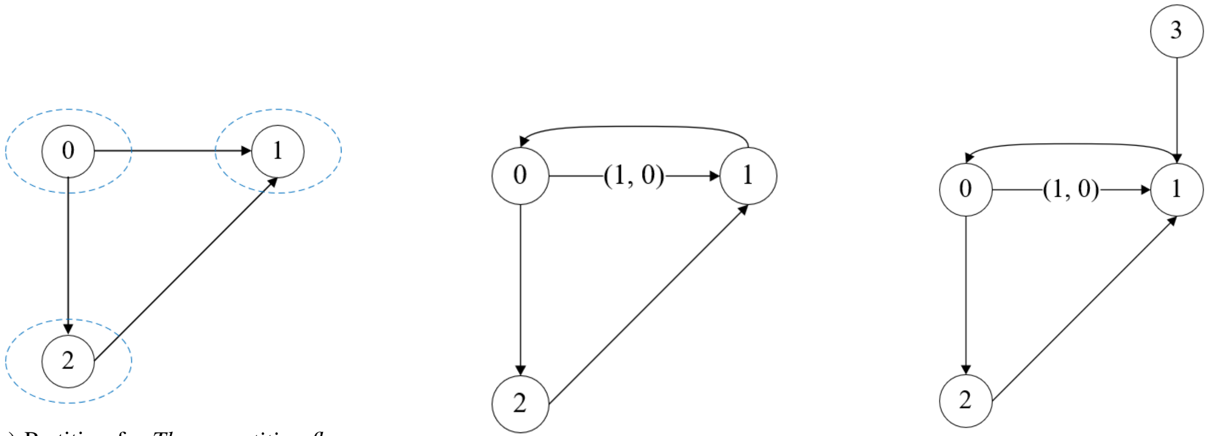
(a) Network structure. (b) Optimal LP solution (z). (c) Optimal LP solution (x).

Figure 8: Example for *Three-partition flow cover inequality*.

3.4.2. Three-partition flow cover inequality with two sources

Symmetrically, we can have *three-partition flow cover inequality with two sources* by using S_1 and S_2 as source nodes, S_0 as destination nodes. For $C_1 \subseteq (S_1, S_0)$, $C_2 \subseteq (S_2, S_0)$, and $C_{21} \subseteq (S_2, S_1)$, the set $C = C_1 \cup C_2 \cup C_{12}$ is a three-partition flow cover if

1. $\lambda_1 = L|C_1| - d_{S_1,S_0} \geq 0$,
2. $\lambda_2 = L|C_2 \cup C_{21}| - d_{S_2,S_0} \geq 0$,
3. $\lambda = L|C_1 \cup C_2| - \left(d_{S_1,S_0} + d_{S_2,S_0}\right) \geq 0$.



(a) Partition for *Three-partition flow cover inequality*. (b) Original network structure. (c) Extended network structure.

Figure 9: Example for three-partition flow cover inequality on extended network.

Furthermore, the three-partition flow cover is minimal if

4. $\lambda_2 < L$ and $\lambda \leq L$.

Given a three-partition flow cover, the *three-partition flow cover inequality with two sources* is defined as

$$\sum_{i=1,2}\sum_{a\in C_i}\left[x_a+\rho_i(1-z_a)\right]-\sum_{i=1,2}\sum_{a\in(S_0,S_i)}\min\left\{x_a,(L-\rho_i)z_a\right\}+\sum_{i=1,2}\sum_{a\in(S_i,S_0)\setminus C_i}\max\left\{x_a-\rho_i z_a,0\right\}$$
$$+\sum_{a\in C_{21}}(\rho_2-\rho_1)(1-z_a)-\sum_{a\in(S_2,S_1)\setminus C_{21}}\min\left\{x_a,(\rho_2-\rho_1)z_a\right\}+\sum_{a\in(S_1,S_2)}\max\left\{0,x_a+(\rho_2-\rho_1\right.$$
$$\left.-L)z_a\right\}\leq d_{S_1,S_0}+d_{S_2,S_0}, \tag{16}$$

where

$$(\rho_1,\rho_2)=\begin{cases}\left(L-\lambda,L-\lambda+(\lambda-\lambda_2)_+\right)\\ \quad\text{(Type 3 three-partition flow cover)}\\ \left((\lambda_2-\lambda)_+,L-\lambda_2+(\lambda_2-\lambda)_+\right)\\ \quad\text{(Type 4 three-partition flow cover)}\end{cases}$$

Proposition 4. *The* three-partition flow cover inequality with two destinations *is valid for the NDCL problem, and strengthens formulation 1.*

Now we look at another example depicted in Figure 9b. Assume three are three commodities, where $s_1 = s_2 = 0$, $d_1 = d_2 = 1$, $s_3 = 1$, $d_3 = 2$. The optimal LP solution is

$$\begin{aligned}x^1_{0,2}=x^1_{2,1}=x^2_{0,2}=x^2_{2,1}&=0.5,\\ x^1_{0,1}=x^2_{0,1}=z_{0,1}&=0.5,\\ z_{1,0}=z_{0,2}=z_{2,1}&=1,\\ x^3_{1,0}=x^3_{0,2}&=1.\end{aligned}$$

It can be checked that no cover inequality is violated. However, if we extend the graph to the network in 9c, and reset $s_3 = 3$. The optimal LP solution will be $x^3_{3,1} = z^3_{3,1} = 1$, all the others are the same. Let $S_0 = \{0, 3\}$, $S_1 = \{1\}$, and $S_2 = \{2\}$, $C_1 = \{(0, 1)\}$, $C_2 = \{(0, 2)\}$, $C_{1,2} = \emptyset$. Then $\lambda_1 0$, $\lambda_2 = 1$, $\lambda = 1$. The type 2 three-partition flow cover inequality will be

$$x_{0,1} + \rho_1(1 - z_{0,1}) + x_{0,2} + \rho_2(1 - z_{0,2}) - \min\left\{x_{1,0}, (L - \rho_1)z_{1,0}\right\} + \max\left\{0, x_{3,1} - \rho_1 z_{3,1}\right\} - \max\left\{0, x_{2,1} + (\rho_2 - \rho_1 - L)z_{2,1}\right\} = 3.5 > 3.$$

The optimal LP solution violates the type 2 three-partition flow cover inequality. This example shows that it's better to separate the source and destination node by augmenting the network when we aggregate the commodities. This example also implies that when our assumption $d_{S_1,S_0} = d_{S_1,S_2} = d_{S_2,S_1} = d_{S_2,S_0} = 0$ is not satisfied, how could we deal with it.

3.5. Incompatible r-arc-commodity Inequalities

The integer flow requirement makes the multicommodity network design problem much more difficult to solve to optimality. In general, linear multicommodity flow problems can have fractional flows, even if all data is integral. As shown in Figure 10, the optimal solution for the relaxed LP problem will be $x^i_{s_i,d_i} = 1/3$, $i = 1, ..., 4$, the total cost is $4/3$. One optimal integer solution could be $x^1_{s_1,d_1} = 1$, $x^2_{s_2,d_2} = 1$, the total cost is 2. Clearly, all the inequalities listed above are satisfied for the optimal LP solution. To cut off the fractional solution here, we introduce the *incompatible r-arc-commodity inequalities*.

Like the *incompatible r-arc inequalities* in Balakrishnan et al. (2017), we observe that when we choose exactly one arc for each commodity, say a_k, not all of them are compatible. Assume at most $r - 1$ of them are compatible, we will have the *incompatible r-arc-commodity Inequalities*,

$$\sum_{k \in K} x^k_{a_k} \le r - 1. \tag{17}$$

This observation implies that, given arc-commodity pairs, we need to find the maximum of compatible ones. The maximum can be found by the following optimization problem,

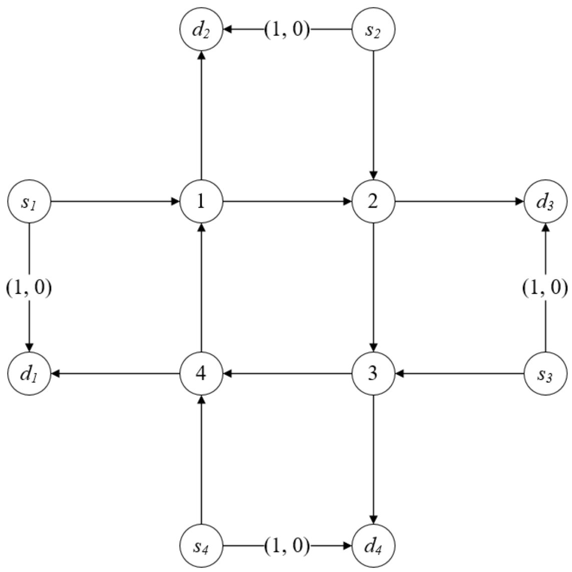
$$\begin{aligned}
\max \; & \sum_{k \in K} x^k_{a_k} \\
\text{s.t.} \; & \\
& \sum_{a \in A^-(i)} x^k_a - \sum_{a \in A^+(i)} x^k_a = \begin{cases} -1 & \text{if } i = s_k, \\ 1 & \text{if } i = d_k, \\ 0 & \text{otherwise,} \end{cases} \\
& \sum_{k \in K} x^k_a \le L z_a, \forall a \in A, \\
& x^k_a \in [0, 1], \forall a \in A, \forall k \in K, \\
& z_a \in [0, 1], \forall a \in A.
\end{aligned} \tag{18}$$

Let r_m be the maximum of compatible arc-commodity pairs, it may not be an integer. Since all $x^k_{a_k}$ are integers, we can reformulate the inequality as

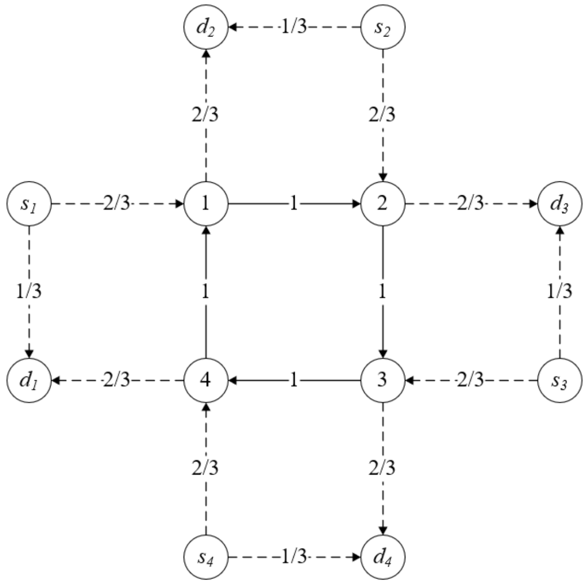
$$\sum_{k \in K} x^k_{a_k} \le \lfloor r_m \rfloor . \tag{19}$$

Proposition 5. *The* incompatible r-arc-commodity inequality *19 is valid for the NDCL problem and can strength formulation [NDCL].*

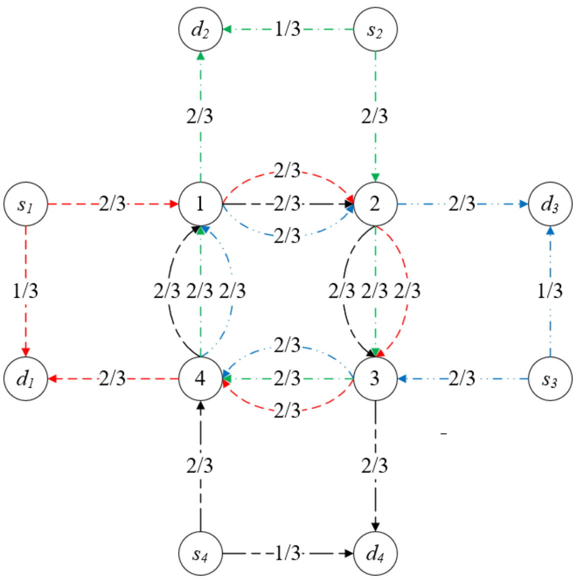
For the example in Fig 10, if we choose arc (s_k, k) for commodity k, we observe that at most two of them are



(a) Network structure.



(b) Optimal LP solution (z).



(c) Optimal LP solution (x).

Figure 10: Example for *r-arc-commodity inequality*.

compatible. Actually, solving the maximization, we get $r_m = 8/3$. The corresponding *r-arc-commodity inequality* is

$$x^1_{s_1,1} + x^2_{s_2,2} + x^3_{s_3,3} + x^4_{s_4,4} \leq \left\lfloor \frac{8}{3} \right\rfloor = 2.$$

Adding *r-arc-commodity inequality* $x^1_{s_1,1} + x^2_{s_2,2} + x^3_{s_3,3} + x^4_{s_4,4} \leq 2$, we can get the optimal integer solution.

4. Conclusions

In this paper, we study the constant capacity multicommodity fixed-charge network design problems. Using polynomial time reduction and the NP-completeness of the three-dimensional matching, we show that the constant capacity multicommodity fixed-charge network design problem is NP-complete even if the arc capacity is uniform over the network and as small as 2. We have developed some inequalities to strengthen the LP relaxation as well as a cut-set generation algorithm based on metaheuristics principles. The constant capacity multicommodity fixed-charge network design problem is closely related to the transportation problem, especially the platooning problem with length constraint. It would be interesting to investigate how to apply the cutting plane algorithm in designing an efficient platooning system for heavy-duty trucks.

References

Andrews, M., 2004. Hardness of buy-at-bulk network design, pp. 115 – 124. doi:10.1109/FOCS.2004.32.

Antonakopoulos, S., 2011. Approximating directed buy-at-bulk network design, in: Jansen, K., Solis-Oba, R. (Eds.), Approximation and Online Algorithms, Springer Berlin Heidelberg, Berlin, Heidelberg. pp. 13–24.

Antonakopoulos, S., Chekuri, C., Shepherd, B., Zhang, L., 2011. Buy-at-bulk network design with protection. Mathematics of Operations Research 36, 71–87. URL: http://www.jstor.org/stable/23011948.

Atamturk, A., 2002. On capacitated network design cut-set polyhedra. Math. Program. 92, 425–437. doi:10.1007/s101070100284.

Atamturk, A., Rajan, D., 2002. On splittable and unsplittable capacitated network design arc-set polyhedra. Math. Program. 92, 315–333. doi:10.1007/s101070100269.

Atamtürk, A., Gómez, A., Küçükyavuz, S., 2016. Three-partition flow cover inequalities for constant capacity fixed-charge network flow problems. Networks 67, 299–315. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/net.21677, doi:https://doi.org/10.1002/net.21677, arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.1002/net.21677.

Balakrishnan, A., Li, G., Mirchandani, P., 2017. Optimal network design with end-to-end service requirements. Operations Research 65, 729–750. URL: https://doi.org/10.1287/opre.2016.1579, doi:10.1287/opre.2016.1579, arXiv:https://doi.org/10.1287/opre.2016.1579.

Chakrabarty, D., Ene, A., Krishnaswamy, R., Panigrahi, D., 2018. Online buy-at-bulk network design. SIAM Journal on Computing 47, 1505–1528. URL: https://doi.org/10.1137/16M1117317, doi:10.1137/16M1117317, arXiv:https://doi.org/10.1137/16M1117317.

Charikar, M., Karagiozova, A., 2005. On non-uniform multicommodity buy-at-bulk network design, in: Proceedings of the Thirty-Seventh Annual ACM Symposium on Theory of Computing, Association for Computing Machinery, New York, NY, USA. p. 176–182. URL: https://doi.org/10.1145/1060590.1060617, doi:10.1145/1060590.1060617.

Chekuri, C., Hajiaghayi, M.T., Kortsarz, G., Salavatipour, M.R., 2007. Approximation algorithms for node-weighted buy-at-bulk network design, in: Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms, Society for Industrial and Applied Mathematics, USA. p. 1265–1274.

Chekuri, C., Hajiaghayi, M.T., Kortsarz, G., Salavatipour, M.R., 2010. Approximation algorithms for nonuniform buy-at-bulk network design. SIAM J. Comput. 39, 1772–1798.

Chouman, M., Crainic, T.G., Gendron, B., 2017. Commodity representations and cut-set-based inequalities for multicommodity capacitated fixed-charge network design. Transportation Science 51, 650–667. URL: https://doi.org/10.1287/trsc.2015.0665, doi:10.1287/trsc.2015.0665, arXiv:https://doi.org/10.1287/trsc.2015.0665.

Costa, A., Cordeau, J.F., Gendron, B., 2009. Benders, metric and cutset inequalities for multicommodity capacitated network design. Computational Optimization and Applications 42, 371–392. doi:10.1007/s10589-007-9122-0.

Crainic, T.G., 2000. Service network design in freight transportation. European Journal of Operational Research 122, 272 – 288. URL: http://www.sciencedirect.com/science/article/pii/S0377221799002337, doi:https://doi.org/10.1016/S0377-2217(99)00233-7.

Frangioni, A., Gendron, B., 2009. 0–1 reformulations of the multicommodity capacitated network design problem. Discrete Applied Mathematics 157, 1229 – 1241. URL: http://www.sciencedirect.com/science/article/pii/S0166218X0800200X, doi:https://doi.org/10.1016/j.dam.2008.04.022. reformulation Techniques and Mathematical Programming.

Gassner, E., 2010. The steiner forest problem revisited. Journal of Discrete Algorithms 8, 154 – 163. URL: http://www.sciencedirect.com/science/article/pii/S1570866709000628, doi:https://doi.org/10.1016/j.jda.2009.05.002. selected papers from the 3rd Algorithms and Complexity in Durham Workshop ACiD 2007.

Goemans, M.X., Williamson, D.P., 1995. A general approximation technique for constrained forest problems. SIAM J. Comput. 24, 296–317. URL: https://doi.org/10.1137/S0097539793242618, doi:10.1137/S0097539793242618.

Gupta, A., Kumar, A., 2015. Greedy algorithms for steiner forest, in: Proceedings of the Forty-Seventh Annual ACM Symposium on Theory of Computing, Association for Computing Machinery, New York, NY, USA. p. 871–878. URL: https://doi.org/10.1145/2746539.2746590, doi:10.1145/2746539.2746590.

Gupta, A., Kumar, A., Roughgarden, T., 2003. Simpler and better approximation algorithms for network design, in: Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing, Association for Computing Machinery, New York, NY, USA. p. 365–372. URL: https://doi.org/10.1145/780542.780597, doi:10.1145/780542.780597.

Herrmann, J., Ioannou, G., Minis, I., Proth, J., 1996. A dual ascent approach to the fixed-charge capacitated network design problem. European Journal of Operational Research 95, 476 – 490. URL: http://www.sciencedirect.com/science/article/pii/0377221795003053, doi:https://doi.org/10.1016/0377-2217(95)00305-3.

Holmberg, K., Yuan, D., 2000. A lagrangian heuristic based branch-and-bound approach for the capacitated network design problem.

Operations Research 48, 461–481. URL: https://doi.org/10.1287/opre.48.3.461.12439, doi:10.1287/opre.48.3.461.12439, arXiv:https://doi.org/10.1287/opre.48.3.461.12439.

Hubert Chan, T.H., Hu, S., Jiang, S.H.C., 2018. A ptas for the steiner forest problem in doubling metrics. SIAM Journal on Computing 47, 1705–1734. URL: https://doi.org/10.1137/16M1107206, doi:10.1137/16M1107206, arXiv:https://doi.org/10.1137/16M1107206.

Karp, R., 1972. Reducibility among combinatorial problems, pp. 85–103. doi:10.1007/978-3-540-68279-0_8.

Katayama, N., Chen, M., Kubo, M., 2009. A capacity scaling heuristic for the multicommodity capacitated network design problem. Journal of Computational and Applied Mathematics 232, 90 – 101. URL: http://www.sciencedirect.com/science/article/pii/S0377042708005487, doi:https://doi.org/10.1016/j.cam.2008.10.055. special Issue: Honor of Professor Hideo Kawarada on the Occasion of his 70th Birthday.

Korte, B., Vygen, J., 2007. Combinatorial Optimization: Theory and Algorithms. 4th ed., Springer Publishing Company, Incorporated.

Magnanti, T.L., Mirchandani, P., Vachani, R., 1993. The convex hull of two core capacitated network design problems. Mathematical Programming 60, 233–250. doi:10.1007/BF01580612.

Magnanti, T.L., Wong, R.T., 1984. Network design and transportation planning: Models and algorithms. Transportation Science 18, 1–55. URL: https://doi.org/10.1287/trsc.18.1.1, doi:10.1287/trsc.18.1.1, arXiv:https://doi.org/10.1287/trsc.18.1.1.

Minoux, M., 1989. Networks synthesis and optimum network design problems: Models, solution methods and applications. Networks 19, 313–360. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/net.3230190305, doi:10.1002/net.3230190305, arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.1002/net.3230190305.

Natu, M., Shu-Cherng, F., 1997. The point-to-point connection problem — analysis and algorithms. Discrete Applied Mathematics 78, 207 – 226. URL: http://www.sciencedirect.com/science/article/pii/S0166218X97000103, doi:https://doi.org/10.1016/S0166-218X(97)00010-3.

Padberg, M.W., Van Roy, T.J., Wolsey, L.A., 1985. Valid linear inequalities for fixed charge problems. Operations Research 33, 842–861. URL: https://doi.org/10.1287/opre.33.4.842, doi:10.1287/opre.33.4.842, arXiv:https://doi.org/10.1287/opre.33.4.842.

Rodríguez-Martín, I., Salazar-González], J.J., 2010. A local branching heuristic for the capacitated fixed-charge network design problem. Computers & Operations Research 37, 575 – 581. URL: http://www.sciencedirect.com/science/article/pii/S0305054808001585, doi:https://doi.org/10.1016/j.cor.2008.09.003. hybrid Metaheuristics.

Salman, F.S., Cheriyan, J., Ravi, R., Subramanian, S., 1997. Buy-at-bulk network design: Approximating the single-sink edge installation problem, in: Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, Society for Industrial and Applied Mathematics, USA. p. 619–628.